LU factorizations, $q = 0$ limits, and p-adic interpretations of some q-hypergeometric orthogonal polynomials

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Dedicated to Dick Askey on the occasion of his seventieth birthday

Abstract

For little q-Jacobi polynomials and q-Hahn polynomials we give particular q-hypergeometric series representations in which the termwise $q = 0$ limit can be taken. When rewritten in matrix form, these series representations can be viewed as LU factorizations. We develop a general theory of LU factorizations related to complete systems of orthogonal polynomials with discrete orthogonality relations which admit a dual system of orthogonal polynomials. For the $q = 0$ orthogonal limit functions we discuss interpretations on p-adic spaces. In the little 0-Jacobi case we also discuss product formulas.

1 Introduction

This paper is concerned with limits for $q \downarrow 0$ of some q-hypergeometric orthogonal polynomials, in particular little q-Jacobi polynomials and q-Hahn polynomials. Limits of q-hypergeometric polynomials as $q \uparrow 1$ are well-known, see [13, Chapter 5]. Many (q-)hypergeometric orthogonal polynomials have interpretations in connection with (quantum) group representations, for instance as spherical or intertwining functions, matrix elements, Clebsch-Gordan coefficients and Racah coefficients, see for instance [29] and [12]. Often, the $q \uparrow 1$ limit of the polynomials corresponds to the $q \uparrow 1$ limit from the quantum group to the classical group.

Limits of q-hypergeometric polynomials for $q \downarrow 0$ have been considered for q-ultraspherical polynomials (see [1, §5]) and for more general Askey-Wilson polynomials (see [4, pp. 26–28] and references given there). The limit functions have interpretations as spherical functions on homogeneous trees (see references in [4, p. 28]) and on infinite distance-transitive graphs (see [30]). Note that homogeneous trees are locally compact but noncompact homogeneous spaces of the group $GL(2, \mathbb{Q}_p)$ ($\mathbb{Q}_p$ the field of p-adic numbers). No geometric explanation of this $q \downarrow 0$ limit is known, see also the discussion in [14]. Macdonald considered the $q \downarrow 0$ limit of Macdonald polynomials, both for root system A_n (yielding Hall-Littlewood polynomials, see [18, Ch. III]) and for general root systems (see [19, §10]). The limit functions have interpretations as spherical functions on a p-adic Lie group (see [17]), in particular in the A_{d-1} case on $GL(d, \mathbb{Q}_p)$ (see [18, Ch. V]).

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Haran [11] considered limits for $q \downarrow 0$ of q-Hahn polynomials, little q-Jacobi polynomials and little q-Laguerre polynomials. He gave interpretations of these limit functions as spherical or intertwining functions on compact p-adic groups; in particular in the *little 0-Jacobi* case on the group $GL(d, \mathbb{Z}_p)$ ($\mathbb{Z}_p$ the ring of p-adic integers). In [21] Haran's ideas were extended to the higher rank case and the importance of the cellular basis, defined in [6], was emphasized. All these interpretations are valid with respect to the larger family of *p-adic fields*, or more generally *non-Archimedean fields*. Altogether, q-special functions clearly play an important role as interpolants between representations of groups over all local fields ($\mathbb{R}$ and $\mathbb{C}$ for $q = 1$, and non-Archimedean local fields for $q = 0$).

The present paper gives new proofs of Haran's [11] $q \downarrow 0$ limit results for little q-Jacobi polynomials and q-Hahn polynomials (sections 2 and 4) by starting with a series representation for these polynomials which can be viewed as writing the square (possibly infinite) matrix corresponding to the orthogonal polynomials as a product of a lower triangular matrix and an upper triangular matrix (an *LU factorization*). The matrix elements of the upper triangular matrix (simply $(q^x; q^{-1})_k$) are the q-analogues of the cellular basis in the rank one case of [6]. This observation was decisive for the second author in order to find the higher rank analogue of this $q \downarrow 0$ limit, see [21]. For the limiting little 0-Jacobi functions we give a product formula. In section 3 we discuss the *LU* factorization more generally for orthogonal polynomials with discrete orthogonality relations which form a complete orthogonal system and for which the dual orthogonal system also consists of orthogonal polynomials. We consider an upper-lower factorization as well, When these systems of orthogonal polynomials are moreover finite (so-called Leonard pairs) then our theory is related to Terwilliger [27]. In section 4 we apply the general theory of section 3 to the little q-Jacobi and the q-Hahn case. Finally, in section 5 we give p-adic group interpretations of the $q = 0$ results obtained in section 2.

In an earlier version [15] of this paper we also discussed big q-Jacobi polynomials. However, this family does not fit nicely into the general theory of this paper, and interpretations on p-adic groups are yet missing.

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2 Little q-Jacobi polynomials

In this section we describe the main themes of this paper through the little q-Jacobi polynomials. These polynomials and their p-adic limit were the pioneering example which motivated this paper. The cellular structure of the p-adic Hecke algebra of Grassmannians of lines (see §5) is essentially an *LU* factorization of the matrix corresponding to the spherical functions w.r.t. a geometric basis. As this basis is the limit of little q-Jacobi polynomials, it was natural to seek for such factorization for the matrix corresponding to the latter polynomials.

Throughout we assume $0 < q < 1$. See standard formulas for little q-Jacobi polynomials in [13, §3.12].

2.1 Limit for $q \downarrow 0$

Little q-Jacobi polynomials are given by

$$p_n(x;a,b;q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) \qquad (n \in \mathbb{Z}_{\geq 0}). \tag{2.1}$$

For $b = 0$ they are known as *little q-Laguerre polynomials* or *Wall polynomials* $p_n(x;a;q) := p_n(x;a,0;q)$, see [13, §3.20]. It will turn out that we have to rescale the parameters a and b in order to be able to take the limit of these polynomials for $q \downarrow 0$. We define:

$$p_n^{a,b;q}(x) := p_n(q^x; q^{-1}a, q^{-1}b; q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n-1} \\ a \end{matrix}; q, q^{x+1}\right) \tag{2.2}$$
$$(n \in \mathbb{Z}_{\geq 0},\ x \in \mathbb{Z}_{\geq 0} \cup \{\infty\},\ 0 < a < 1,\ b < 1).$$

By [13, (3.12.2)] the functions (2.2) satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} p_m^{a,b;q}(x)\, p_n^{a,b;q}(x)\, w_x^{a,b;q} = \frac{\delta_{m,n}}{\omega_n^{a,b;q}} \qquad (m,n \in \mathbb{Z}_{\geq 0}), \tag{2.3}$$

where

$$w_x^{a,b;q} := \frac{(a;q)_\infty}{(ab;q)_\infty} \frac{(b;q)_x}{(q;q)_x} a^x, \tag{2.4}$$

$$\omega_n^{a,b;q} := \frac{1-abq^{2n-1}}{1-abq^{n-1}} \frac{(a,ab;q)_n}{a^n\,(q,b;q)_n}. \tag{2.5}$$

Note that the weights $w_x^{a,b;q}$ and the dual weights $\omega_n^{a,b;q}$ are positive under the constraints for a and b given in (2.2). Since the little q-Jacobi are orthogonal polynomials with respect to an orthogonality measure of bounded support, they form a complete orthogonal system in the L^2 space with respect to this measure, so the functions (2.2) also form a complete orthogonal system in $\ell^2(\mathbb{Z}_{\geq 0}; w^{a,b;q})$.

As was observed in [24, (4.1), (4.2)], [10, Remark 3.1] and [5, (5.1), (5.3)], little q-Jacobi polynomials can alternatively be expressed as a terminating ${}_3\phi_1$ by application of the transformation formula [9, (III.8) or Exercise 1.15 (ii)] to the terminating ${}_2\phi_1$ in (2.2). We obtain:

$$p_n^{a,b;q}(x) = q^{\frac{1}{2}n(n-1)}\,(-a)^n\,\frac{(b;q)_n}{(a;q)_n}\,{}_3\phi_1\left(\begin{matrix} q^{-n}, abq^{n-1}, q^{-x} \\ b \end{matrix}; q, \frac{q^{x+1}}{a}\right). \tag{2.6}$$

As was also observed in the papers just quoted, this is related to the fact that little q-Jacobi polynomials are the duals of *q^{-1}-Al-Salam Chihara polynomials*

$$Q_n\left(\tfrac{1}{2}(aq^{-x} + a^{-1}q^x); a, b \mid q^{-1}\right)$$
$$= (-1)^n b^n q^{-\frac{1}{2}n(n-1)}\,((ab)^{-1};q)_n\,{}_3\phi_1\left(\begin{matrix} q^{-n}, q^{-x}, a^{-2}q^x \\ (ab)^{-1} \end{matrix}; q, q^n ab^{-1}\right). \tag{2.7}$$

From (2.6) and (2.7) we get indeed the duality

$$
\begin{aligned}
(-a)^{-n}\, q^{-\frac{1}{2}n(n-1)}\, \frac{(a;q)_n}{(b;q)_n}\, p_n^{a,b;q}(x) &= \frac{(-1)^x (q^{-1}a^{-1}b)^{\frac{1}{2}x} q^{\frac{1}{2}x(x-1)}}{(b;q)_x} \\
&\quad \times Q_x\left(\tfrac{1}{2}(qab)^{-\frac{1}{2}}(q^{-n}+abq^{n-1}); (qab)^{-\frac{1}{2}}, (qab^{-1})^{\frac{1}{2}}; q^{-1}\right) \quad (n,x\in\mathbb{Z}_{\ge 0}) \qquad (2.8)
\end{aligned}
$$

Formula (2.6) can be rewritten as:

$$
p_n^{a,b;q}(x) = \sum_{k=0}^{\min(n,x)} q^{\frac{1}{2}(n-k)(n-k-1)}\, (-a)^{n-k}\, \frac{(bq^k;q)_{n-k}\,(abq^{n-1};q)_k}{(a;q)_n\,(q;q)_k}\, (q^n,q^x;q^{-1})_k \quad (x\in\mathbb{Z}_{\ge 0}). \qquad (2.9)
$$

We obtain as an immediate corollary of (2.9):

Theorem 2.1. *The limit functions* (little 0-Jacobi functions)

$$
p_n^{a,b;0}(x) := \lim_{q\downarrow 0} p_n^{a,b;q}(x) \quad (x\in\mathbb{Z}_{\ge 0}) \qquad (2.10)
$$

exist. They are equal to

$$
p_0^{a,b;0}(x) = 1, \qquad (2.11)
$$

$$
p_1^{a,b;0}(x) = \begin{cases} -\dfrac{a(1-b)}{1-a} & \text{if } x=0, \\ 1 & \text{if } x>0, \end{cases} \qquad (2.12)
$$

$$
p_n^{a,b;0}(x) = \left\{ \begin{array}{cl} 0 & \text{if } 0\le x<n-1, \\ -\dfrac{a}{1-a} & \text{if } x=n-1, \\ 1 & \text{if } x>n-1 \end{array} \right\} \quad (n\ge 2). \qquad (2.13)
$$

Remark 2.2. Theorem 2.1 was first stated by Haran [11, (7.3.37)], where the limit functions (2.11)–(2.13) are given in [11, (4.4.9)]. The limit result there follows from the expression for little q-Jacobi polynomials on [11, p.59, second formula from below], which reads in our notation as:

$$
\begin{aligned}
p_n^{a,b;q}(x) &= q^{nx}\, \frac{(b;q)_n}{(a^{-1}q^{1-n};q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, a^{-1}q^{1-n}, q^{-x} \\ b, 0 \end{matrix}; q, q\right) \qquad (2.14) \\
&= \sum_{k=0}^{\min(n,x)} q^{\frac{1}{2}(n-k)(n+2x-3k-1)}\, \frac{(-a)^{n-k}\,(bq^k;q)_{n-k}}{(a;q)_{n-k}\,(q;q)_k}\, (q^n;q^{-1})_k\,(q^x;q^{-1})_k\,. \qquad (2.15)
\end{aligned}
$$

Formula (2.14) follows from (2.2) by the transformation formula [9, (1.5.6)]. Theorem 2.1 can be obtained from (2.15) by letting $q\downarrow 0$.

Haran's [11, Ch. 7] notation is connected with ours by:

$$\frac{\phi_{q,n}^{(\alpha)\beta}(g^x)}{\phi_{q,n}^{(\alpha)\beta}(0)} = p_n^{q^\beta,q^\alpha;q}(x), \qquad \zeta_{(q)}(s) = ((q^s;q)_\infty)^{-1}. \tag{2.16}$$

Haran [11, pp. 61–64] also considers the little q-Laguerre case ($\alpha \to \infty$ in (2.16); $b = 0$ everywhere in our notation) and its $q = 0$ limit.

Remark 2.3. From (2.9) we also obtain the following asymptotics of $p_n^{a,b;q}(x)$ as $q \downarrow 0$:

$$\lim_{q\downarrow 0} q^{-\frac{1}{2}(n-x)(n-x-1)}\, p_n^{a,b;q}(x) = \frac{(-a)^{n-x}}{1-a} \qquad (1 \le x \le n-1). \tag{2.17}$$

Alternatively, (2.17) can be derived from the q-difference equation [13, (3.12.5)] by induction with respect to x, starting at $x = 1$. Theorem 2.1 can also be proved by use of (2.17).

From (2.4) and (2.5) we get limits

$$w_x^{a,b;0} := \lim_{q\downarrow 0} w_x^{a,b;q} = \begin{cases} \dfrac{1-a}{1-ab} & \text{if } x = 0, \\ \dfrac{(1-a)(1-b)}{1-ab}\, a^x & \text{if } x > 0, \end{cases} \tag{2.18}$$

$$\omega_n^{a,b;0} := \lim_{q\downarrow 0} \omega_n^{a,b;q} = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{1-a}{a(1-b)} & \text{if } n = 1, \\ \dfrac{(1-a)(1-ab)}{a^n(1-b)} & \text{if } n > 1. \end{cases} \tag{2.19}$$

Note that $w_x^{a,b;0} > 0$ ($x \in \mathbb{Z}_{\ge 0}$) and $\omega_n^{a,b;0} > 0$ ($n \in \mathbb{Z}_{\ge 0}$) if $0 < a < 1$ and $b < 1$. The orthogonality relation (2.3) remains valid for $q = 0$, as can be verified by use of (2.11)–(2.13) and (2.18)–(2.19). Formally, we can obtain the case $q = 0$ of (2.3) by taking termwise limits.

2.2 *LU* factorization

Formula (2.9) has the big advantage over (2.15) that it can be rewritten in matrix form as a product of a lower and an upper triangular matrix:

$$P^q = L^q\, U^q, \quad \text{i.e.,} \quad P_{n,x}^q = \sum_{k=0}^{\infty} L_{n,k}^q\, U_{k,x}^q = \sum_{k=0}^{\min(n,x)} L_{n,k}^q\, U_{k,x}^q \qquad (n, x \in \mathbb{Z}_{\ge 0}), \tag{2.20}$$

where

$$P_{n,x}^q := p_n^{a,b;q}(x), \qquad L_{n,k}^q := q^{\frac{1}{2}(n-k)(n-k-1)}\, (-a)^{n-k}\, \frac{(bq^k;q)_{n-k}\, (abq^{n-1};q)_k}{(a;q)_n\, (q;q)_k}\, (q^n;q^{-1})_k\,,$$
$$U_{k,x}^q := (q^x;q^{-1})_k\,. \tag{2.21}$$

As a limit case of (2.20), (2.21) formulas (2.11)–(2.13) can similarly be rewritten in matrix form (2.20) with $q = 0$, where

$$
\begin{aligned}
P^0_{n,x} &:= p^{a,b;0}_n(x), \\
U^0_{k,x} &:= \begin{cases} 1 & \text{if } k \le x, \\ 0 & \text{if } k > x, \end{cases} \\
L^0_{n,k} &:= \quad 0 \quad \text{if } \ k \ne n \ \text{ or } \ n-1, \\
L^0_{n,n-1} &:= \begin{cases} -\dfrac{a(1-b)}{1-a} & \text{if } n = 1, \\ -\dfrac{a}{1-a} & \text{if } n > 1, \end{cases} \\
L^0_{n,n} &:= \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{1-ab}{1-a} & \text{if } n = 1, \\ \dfrac{1}{1-a} & \text{if } n > 1. \end{cases}
\end{aligned}
\tag{2.22}
$$

Again, L^0 is a lower triangular and U^0 an upper triangular matrix. The functions $c^q_k(q^x) := U^q_{k,x}/U^q_{k,k} = (q^x; q^{-1})_k/(q;q)_k$ and $c^0_k(x) := U^0_{k,x}$ (see (2.21) and (2.22)) can be considered as forming a *cellular basis* in the terminology of [6, §3.3] and [21]. The functions $q^x \to \text{const.}\, U^q_{k,x}$ can also be considered as the one-variable cases of Okounkov's [20] *shifted Macdonald polynomials.*

Remark 2.4. For $0 \le q < 1$ we can consider P^q as the matrix of a unitary operator from the Hilbert space $\ell^2(\mathbb{Z}_{\ge 0}; (w^q)^{-1})$ onto the Hilbert space $\ell^2(\mathbb{Z}_{\ge 0}; \omega^q)$. Let $v_k := a^{-k}$. Probably, U^q is the matrix of a bounded linear operator from $\ell^2(\mathbb{Z}_{\ge 0}; (w^q)^{-1})$ to $\ell^2(\mathbb{Z}_{\ge 0}; v)$, and L^q is the matrix of a bounded linear operator from $\ell^2(\mathbb{Z}_{\ge 0}; v)$ to $\ell^2(\mathbb{Z}_{\ge 0}; \omega^q)$. Probably, the operators corresponding to U^q and L^q have bounded inverses with matrices given by the explicit matrix inverses of U^q and L^q.

2.3 A product formula for little 0-Jacobi functions

Theorem 2.5. *The functions $p^{a,b;0}_n$ satisfy the product formula*

$$
p^{a,b;0}_n(x)\, p^{a,b;0}_n(y) = \sum_{z=0}^{\infty} c^{a,b,0}_{x,y,z}\, p^{a,b;0}_n(z), \tag{2.23}
$$

where $c^{a,b,0}_{x,y,z}$ *is given by*

$$c^{a,b,0}_{x,y,z} = \begin{cases} \delta_{z,\min(x,y)} & \text{if } x \neq y, \\ \dfrac{1-2a+ab}{1-a} & \text{if } x = y = z = 0, \\ (1-b)a^z & \text{if } 0 = x = y < z, \\ 0 & \text{if } x = y > z, \\ \dfrac{1-2a}{1-a} & \text{if } x = y = z > 0, \\ a^{z-x} & \text{if } 0 < x = y < z. \end{cases} \tag{2.24}$$

In particular,

$$p^{a,b;0}_n(x)\, p^{a,b;0}_n(y) = p^{a,b;0}_n(\min(x,y)) \quad \text{if } x \neq y. \tag{2.25}$$

Under the constraints $0 < a < 1$, $b < 1$ *we have* $c^{a,b,0}_{x,y,z} \geq 0$ *for all* $x, y, z \in \mathbb{Z}_{\geq 0}$ *iff*

$$0 < a \leq \tfrac{1}{2} \quad \text{and} \quad 2 - a^{-1} \leq b < 1. \tag{2.26}$$

Proof Straightforward verification by (2.11)–(2.13). ☐

Corollary 2.6. *The functions* $p^{a,b;0}_n$ *satisfy the product formula*

$$p^{a,b;0}_n(x)\, p^{a,b;0}_n(y) = \frac{1-ab}{1-a} \sum_{z=0}^{\infty} C^{a,b,0}_{x,y,z}\, w^{a,b,0}_z\, p^{a,b;0}_n(z), \tag{2.27}$$

where

$$C^{a,b,0}_{x,y,z} = \frac{1-a}{1-ab} \sum_{n=0}^{\min(x,y,z)+1} p^{a,b,0}_n(x)\, p^{a,b,0}_n(y)\, p^{a,b,0}_n(z)\, \omega^{a,b,0}_n. \tag{2.28}$$

is symmetric in x, y, z*, and for* $x \leq y \leq z$ *explicitly given by*

$$C^{a,b,0}_{x,y,z} = \begin{cases} 0 & \text{if } 0 \leq x < y \leq z, \\ 1 & \text{if } 0 = x = y < z, \\ (1-b)^{-1}a^{-x} & \text{if } 0 < x = y < z, \\ \dfrac{1-2a+ab}{1-a} & \text{if } 0 = x = y = z, \\ \dfrac{1-2a}{(1-a)(1-b)}\, a^{-x} & \text{if } 0 < x = y = z. \end{cases} \tag{2.29}$$

Remark 2.7. Dunkl and Ramirez [8] obtained the little 0-Laguerre functions $p^{a,b;0}_n$ for $(a,b) = (p^{-1}, 0)$ as spherical functions on the ring of p-adic integers, and they derived the above product formula for those special parameter values from that interpretation as spherical functions.

3 LU factorizations: the general case

3.1 Lower times upper

Let us put the results of §2.2 in a more general framework. Let $\mathcal{N} := \{0, 1, \ldots, N\}$ or $\mathbb{Z}_{\geq 0}$ and let $Y := \{y_x\}_{x \in \mathcal{N}}$ be a countable subset of $\mathbb{R}$. Let $\{p_n\}_{n \in \mathcal{N}}$ be a complete system of orthogonal polynomials on Y with respect to positive weights w_x on the points y_x:

$$\sum_{x \in \mathcal{N}} p_n(y_x)\, p_m(y_x)\, w_x = (\omega_n)^{-1}\, \delta_{n,m} \quad (n, m \in \mathcal{N}), \tag{3.1}$$

where $\omega_n > 0$ for all $n \in \mathcal{N}$ and where, by completeness, we also have the dual orthogonality relation

$$\sum_{n \in \mathcal{N}} p_n(y_x)\, p_n(y_{x'})\, \omega_n = (w_x)^{-1}\, \delta_{x,x'} \quad (x, x' \in \mathcal{N}). \tag{3.2}$$

Note that completeness will certainly hold if Y is finite or bounded in $\mathbb{R}$.

We will now introduce some square matrices with row and column indices running over $\mathcal{N}$. Let P be the matrix with entries $P_{n,x} := p_n(y_x)$ $(n, x \in \mathcal{N})$. Let also W be the diagonal matrix with diagonal entries w_x $(x \in \mathcal{N})$ and let Ω be the diagonal matrix with diagonal entries ω_n $(n \in \mathcal{N})$. Then (3.1) and (3.2) can be written in matrix form as, respectively,

$$PWP^t = \Omega^{-1}, \quad P^t \Omega P = W^{-1}. \tag{3.3}$$

Define polynomials c_k of degree k by

$$c_k(y) := \prod_{j=0}^{k-1} (y_j - y) \quad (k \in \mathcal{N},\ y \in \mathbb{R}), \tag{3.4}$$

and let C be the matrix with entries given by

$$C_{k,x} := c_k(y_x) = \prod_{j=0}^{k-1} (y_j - y_x) \quad (k, x \in \mathcal{N}). \tag{3.5}$$

Then $C_{k,x} = 0$ if $k > x$, so C is an upper triangular matrix. Then, for certain unique coefficients $B_{n,k}$ with $B_{n,n} \neq 0$ we have:

$$p_n(y) = \sum_{k=0}^{n} B_{n,k} c_k(y) \quad (n \in \mathcal{N},\ y \in \mathbb{R}), \tag{3.6}$$

$$P_{n,x} = \sum_{k=0}^{\min(n,x)} B_{n,k} C_{k,x} \quad (n, x \in \mathcal{N}), \tag{3.7}$$

$$P = BC, \tag{3.8}$$

where B is the lower triangular matrix corresponding to the coefficients $B_{n,k}$ $(n \geq k)$.

Both B and C have two-sided inverses because they are triangular matrices with nonzero diagonal entries. Furthermore, if $P = B'C'$ is another factorization of P with B' lower triangular and C' upper triangular, then $B' = BD$, $C' = D^{-1}C$ for some invertible diagonal matrix D.

Things become even nicer if we know that there exist orthogonal polynomials dual to $\{p_n\}$, i.e., if there exist polynomials r_x of degree x $(x \in \mathcal{N})$ and a subset $\{z_n\}_{n\in\mathcal{N}}$ of $\mathbb{R}$ such that

$$p_n(y_x) = r_x(z_n) \quad (n, x \in \mathcal{N}). \tag{3.9}$$

Then the polynomials r_x $(x \in \mathcal{N})$ will form a complete system of orthogonal polynomials with respect to the weights ω_n on the points z_n $(n \in \mathcal{N})$. Thus we can apply the previous result to the r_x. Put

$$B_{n,k} := \prod_{i=0}^{k-1}(z_i - z_n) \quad (n, k \in \mathcal{N}). \tag{3.10}$$

Then B is a lower triangular matrix and for some upper triangular matrix C' we have

$$p_n(y_x) = r_x(z_n) = \sum_{k=0}^{\min(n,x)} B_{n,k}C'_{k,x} \quad (n, x \in \mathcal{N}). \tag{3.11}$$

Then $P = BC'$. Hence

$$P = BDC \tag{3.12}$$

for some diagonal matrix D with nonzero diagonal entries δ_k. Hence

$$p_n(y_x) = \sum_{k=0}^{\min(n,x)} \delta_k \prod_{i=0}^{k-1}(z_i - z_n) \prod_{j=0}^{k-1}(y_j - y_x). \tag{3.13}$$

3.2 Inverting the matrices B and C and computing δ_k

The following theorem is the special case $a_j = 1$, $b_j = 0$ of the Theorem in [16, p.48], and it is also the case $f(x) := (x_1 - x)\ldots(x_{m-1} - x)$ $(m \le n)$ of [22, p.54, Exercise 97] (we thank Michael Schlosser for this reference), but we will give here an independent proof.

Theorem 3.1. *For distinct complex numbers y_n $(n \in \mathcal{N})$ let $C = (C_{m,n})_{m,n\in\mathcal{N}}$ be an upper triangular matrix given by* (3.5)*. Then*

$$(C^{-1})_{k,n} = \prod_{\substack{j=0\\ j\ne k}}^{n}(y_j - y_k)^{-1} \quad (0 \le k \le n). \tag{3.14}$$

Proof Let $m \le n$. Put $V_{m,n} := \prod_{m\le i<j\le n}(y_j - y_i)$. We have to show that

$$\sum_{k=m}^{n}\prod_{j=0}^{m-1}(y_j - y_k)\prod_{\substack{j=0\\ j\ne k}}^{n}(y_j - y_k)^{-1} = \delta_{m,n}. \tag{3.15}$$

This is clearly true for $m = n$. For $m < n$ the left-hand side of (3.15) can be rewritten as

$$\begin{aligned}
&\sum_{k=m}^{n} \prod_{\substack{j=m \\ j\neq k}}^{n} (y_j - y_k)^{-1} \\
&= (V_{m,n})^{-1} \sum_{k=m}^{n} (-1)^{k-m} \prod_{\substack{m\le i<j\le n \\ i,j\neq k}} (y_j - y_i) \\
&= (V_{m,n})^{-1} \sum_{k=m}^{n} (-1)^{k-n} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ y_m & \dots & y_{k-1} & y_{k+1} & \dots & y_n \\ \vdots & & \vdots & \vdots & & \vdots \\ y_m^{n-m-1} & \dots & y_{k-1}^{n-m-1} & y_{k+1}^{n-m-1} & \dots & y_n^{n-m-1} \end{vmatrix} \\
&= (V_{m,n})^{-1} \begin{vmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ y_m & \dots & y_n \\ \vdots & & \vdots \\ y_m^{n-m-1} & \dots & y_n^{n-m-1} \end{vmatrix} = 0. \quad \square
\end{aligned}$$

It follows by taking transpose in (3.14) that the inverse of the lower triangular matrix B given by (3.10) is the lower triangular matrix B^{-1} with

$$(B^{-1})_{m,n} = \prod_{\substack{i=0 \\ i\neq n}}^{m} (z_i - z_n)^{-1} \qquad (0 \le n \le m). \tag{3.16}$$

Put $y_\nu := y_N$ if $\mathcal{N} = \{0, 1, \dots, N\}$ and put $y_\nu := \lim_{x\to\infty} y_x$ if $\mathcal{N} = \mathbb{Z}_{\ge 0}$ and if the (finite) limit y_ν exists and is not equal to any y_x $(x \in \mathcal{N})$. We will now derive an explicit expression for the coefficients δ_k in (3.13) involving $p_n(y_\nu)$ for all $n \in \mathcal{N}$. This expression will have practical usage if a simple explicit expression for $p_n(y_\nu)$ is known, as is the case in most examples.

It follows from (3.13) that

$$p_n(y_\nu) = \sum_{k=0}^{n} \delta_k \prod_{i=0}^{k-1} (z_i - z_n) \prod_{j=0}^{k-1} (y_j - y_\nu) = \sum_{k=0}^{n} B_{n,k}\, \delta_k \prod_{j=0}^{k-1} (y_j - y_\nu).$$

Hence, by matrix inversion,

$$\sum_{n=0}^{m} \left(B^{-1}\right)_{m,n} p_n(y_\nu) = \delta_m \prod_{j=0}^{m-1} (y_j - y_\nu).$$

Thus, by (3.16) we obtain the following formula for δ_m:

$$\delta_m = \frac{1}{\prod_{j=0}^{m-1} (y_j - y_\nu)} \sum_{n=0}^{m} p_n(y_\nu) \prod_{\substack{i=0 \\ i\neq n}}^{m} (z_i - z_n)^{-1}. \tag{3.17}$$

3.3 Upper times lower

We obtain from (3.3) that $P = \Omega^{-1}(P^t)^{-1}W^{-1}$ and from (3.12) (only formally in the infinite dimensional case) that $(P^t)^{-1} = (B^t)^{-1}D^{-1}(C^t)^{-1}$. Hence,

$$P = \Omega^{-1}(B^t)^{-1}D^{-1}(C^{-1})^t W^{-1}. \tag{3.18}$$

From Theorem 3.1,(3.10) and (3.5) we see that

$$\big((B^t)^{-1}\big)_{n,k} = \prod_{\substack{i=0\\ i\neq n}}^{k} (z_i - z_n)^{-1} \qquad (0 \le n \le k), \tag{3.19}$$

$$\big(C^{-1}\big)_{x,k} = \prod_{\substack{j=0\\ j\neq x}}^{k} (y_j - y_x)^{-1} \qquad (0 \le x \le k). \tag{3.20}$$

When we substitute everything in (3.18) then we obtain

$$p_n(y_x) = \omega_n^{-1} w_x^{-1} \sum_{k \ge \max(n,x)} \delta_k^{-1} \prod_{\substack{i=0\\ i\neq n}}^{k} (z_i - z_n)^{-1} \prod_{\substack{j=0\\ j\neq x}}^{k} (y_j - y_x)^{-1}. \tag{3.21}$$

Remark 3.2. The derivation of (3.18) is purely formal if $\mathcal{N} = \mathbb{Z}_{\ge 0}$, since we do not know in general if the matrices B and C correspond to bounded linear operators and if these operators have bounded inverses. See [2] and [3] and references given there for some generalities about existence of LU-factorizations of bounded linear operators as a product of a lower triangular and an upper triangular matrix, both corresponding to bounded linear operators. See [31] for an example of a unitary operator on $\ell^2(\mathbb{Z}_{\ge 0})$ without LU-factorization.

4 LU factorizations: examples

4.1 Little q-Jacobi

From (2.2) and (2.8) we see that (3.13) will have meaning with the following substitutions:

$$y_x := q^x, \quad z_i := q^{-i} + abq^{i-1} \quad (x, i \in \mathbb{Z}_{\ge 0}), \qquad \frac{p_n(y_x)}{p_n(y_\nu)} := p_n^{a,b;q}(x), \tag{4.1}$$

where

$$y_\nu = \lim_{x\to\infty} y_x = 0, \qquad p_n(y_\nu) = (-a)^{-n}\, q^{-\frac{1}{2}n(n-1)} \,\frac{(a;q)_n}{(b;q)_n}. \tag{4.2}$$

Rewrite (3.13) as

$$\frac{p_n(y_x)}{p_n(y_\nu)} = \sum_{k=0}^{\min(n,x)} (p_n(y_\nu))^{-1} \left(\prod_{i=0}^{k-1}(z_i - z_n)\right) \delta_k \left(\prod_{j=0}^{k-1} y_j\right) \left(\prod_{j=0}^{k-1}(1 - y_j^{-1} y_x)\right) \tag{4.3}$$

and compare with (2.20), (2.21). We obtain that

$$U^q_{k,x} = \prod_{j=0}^{k-1}(1 - y_j^{-1} y_x), \quad L^q_{n,k} = (p_n(y_\nu))^{-1} \left(\prod_{i=0}^{k-1}(z_i - z_n)\right) \delta_k \left(\prod_{j=0}^{k-1} y_j\right).$$

Hence, by (2.21), (4.1) and (4.2),

$$\delta_m = \frac{(qa^{-1})^m}{(b;q)_m\,(q;q)_m}. \tag{4.4}$$

We can alternatively compute δ_m from (3.17), which takes after substitution of (4.1) and (4.2) the form

$$\delta_m = \frac{q^m}{(ab,q;q)_m}\,{}_6\phi_4\left(\begin{matrix} q^{-1}ab, (qab)^{\frac12}, -(qab)^{\frac12}, a, 0, q^{-m} \\ (q^{-1}ab)^{\frac12}, -(q^{-1}ab)^{\frac12}, b, abq^m \end{matrix}; q, q^{-1}a^m\right). \tag{4.5}$$

The ${}_6\phi_4$ can be evaluated as a confluent limit case of the summation formula for a terminating very well-poised ${}_6\phi_5$ series given in [9, (2.4.2)]. The resulting explicit formula for δ_m by (4.5) coincides with (4.4).

Note that (3.13) with substitution of (4.1) and (4.2) does not immediately allow to take limits for $q \downarrow 0$. For this we have to renormalize $p_n(y_x)$ by division by $p_n(y_\nu)$ and we have to transfer some factors only depending on k from $C_{k,x}$ to $B_{n,k}\delta_k/p_n(y_\nu)$.

Next we consider formula (3.21) (the upper times lower factorization) with substitution of (4.1) and (4.2). Then w_x and ω_n in (3.21) become

$$w_x = w_x^{a,b;q}, \qquad \omega_n := (p_n(y_\nu))^2\,\omega_n^{a,b;q}.$$

With these substitutions, (3.21) can be written in the form

$$p_n^{a,b;q}(x) = \sum_{k=\max(n,x)}^{\infty} q^{\frac12(k-x)(k-x-1)}\,(-a)^{k-x}\,\frac{(abq^{n+k};q)_\infty\,(b;q)_k}{(a;q)_\infty\,(b;q)_x\,(q;q)_k}\,(q^k;q^{-1})_n\,(q^k;q^{-1})_x \tag{4.6}$$

$$= \begin{cases} q^{\frac12(n-x)(n-x-1)}\,(-a)^{n-x}\,\dfrac{(abq^{2n};q)_\infty\,(b;q)_n\,(q^n;q^{-1})_x}{(a;q)_\infty\,(b;q)_x} \\ \qquad \times\,{}_2\phi_2\left(\begin{matrix} q^{n+1}, bq^n \\ q^{n-x+1}, abq^{2n} \end{matrix}; q, q^{n-x}a\right) & \text{if } n \ge x, \\ \dfrac{(abq^{n+x};q)_\infty\,(q^x;q^{-1})_n}{(a;q)_\infty}\,{}_2\phi_2\left(\begin{matrix} bq^x, q^{x+1} \\ q^{x-n+1}, abq^{n+x} \end{matrix}; q, a\right) & \text{if } n \le x. \end{cases} \tag{4.7}$$

Above we started with (3.21). Instead we might have started with (3.18) and then obtain for C and B ((3.5) and (3.10) with substitution of (4.1)) the inverse matrices by [7, (4.2) and (4.11)].

Since (3.21) was only derived in a formal way, we have not yet proved now (4.6) and (4.7) in a rigorous way. However, the expressions (4.7) can be alternatively obtained from (2.6) by first

applying [9, (III.8) or Exercise 1.15 (ii)] and next inverting the series (see [9, Exercise 1.4(ii)]):

$$p_n^{a,b;q}(x) = q^{\frac{1}{2}n(n-1)}\,(-a)^n\,\frac{(bq^x;q)_n}{(a;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ b^{-1}q^{1-n-x} \end{matrix}; q, q^{2-n}(ab)^{-1}\right) \tag{4.8}$$

$$= \begin{cases} q^{\frac{1}{2}(n-x)(n-x-1)}\,(-a)^{n-x}\,\dfrac{(b;q)_n\,(q^n;q^{-1})_x}{(a;q)_n\,(b;q)_x}\,{}_2\phi_1\left(\begin{matrix} bq^n, q^{-x} \\ q^{n-x+1} \end{matrix}; q, aq^n\right) & \text{if } n \ge x, \\ \dfrac{(q^x;q^{-1})_n}{(a;q)_n}\,{}_2\phi_1\left(\begin{matrix} bq^x, q^{-n} \\ q^{x-n+1} \end{matrix}; q, aq^n\right) & \text{if } n \le x. \end{cases} \tag{4.9}$$

Then the expressions (4.9) yield expressions (4.7) by means of [9, (1.5.4)].

4.2 q-Hahn

See standard formulas for *q-Hahn polynomials* in [13, §3.6]. They are given by

$$Q_n(x;a,b,N;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{matrix}; q, q\right) \qquad (N \in \mathbb{Z}_{>0},\ n \in \{0,1,\ldots,N\}). \tag{4.10}$$

It will turn out that we have to renormalize the parameters a and b in order to be able to take the limit of these polynomials for $q \downarrow 0$. For the same reason, we have to consider these polynomials with argument q^{x-N}, rather than the usual argument q^{-x}. We define:

$$Q_n^{a,b,N;q}(x) := Q_n(q^{x-N};q^{-1}a,q^{-1}b,N;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n-1}, q^{x-N} \\ a, q^{-N} \end{matrix}; q, q\right) \tag{4.11}$$
$$(N \in \mathbb{Z}_{>0},\ n,x \in \{0,1,\ldots,N\};\ 0<a<1,\ b<1 \text{ or } a,b>q^{1-N}).$$

By [13, (3.6.2)] the functions (4.11) satisfy the orthogonality relation

$$\sum_{x=0}^{N} Q_m^{a,b,N;q}(x)\,Q_n^{a,b,N;q}(x)\,w_x^{a,b,N;q} = \frac{\delta_{m,n}}{\omega_n^{a,b,N;q}} \qquad (m,n=0,1,\ldots,N), \tag{4.12}$$

where

$$w_x^{a,b,N;q} := \frac{(a;q)_N}{(ab;q)_N}\,\frac{(q^N;q^{-1})_x}{(aq^{N-1};q^{-1})_x}\,\frac{(b;q)_x}{(q;q)_x}\,a^x, \tag{4.13}$$

$$\omega_n^{a,b,N;q} := \frac{(q^N;q^{-1})_n}{(abq^N;q)_n}\,\frac{1-abq^{2n-1}}{1-abq^{n-1}}\,\frac{(a,ab;q)_n}{a^n\,(q,b;q)_n}. \tag{4.14}$$

Note that the weights $w_x^{a,b;q}$ and the dual weights $\omega_n^{a,b;q}$ are positive under the constraints for a and b given in (4.11).

If we apply the transformation formula [9, (3.2.2)] to the terminating ${}_3\phi_2$ in (4.11) then we obtain:

$$
\begin{aligned}
q^{-\frac{1}{2}n(n-1)}\,(-a)^{-n}\,\frac{(a;q)_n}{(b;q)_n}\,Q_n^{a,b,N;q}(x) &= {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n-1}, q^{-x} \\ b, q^{-N}\end{matrix}; q, \frac{q^{x-N+1}}{a}\right) && (4.15)\\
&= {}_3\phi_2\left(\begin{matrix} q^{x}, q^{n}, a^{-1}b^{-1}q^{-n+1} \\ b^{-1}, q^{N}\end{matrix}; q^{-1}, q^{-1}\right). && (4.16)\\
&= Q_n^{b^{-1},a^{-1};q^{-1}}(N-x) && (4.17)\\
&= Q_n(q^x; qb^{-1}, qa^{-1}, N; q^{-1}) && (4.18)\\
&= R_x(q^n + a^{-1}b^{-1}q^{-n+1}; qb^{-1}, qa^{-1}, N \mid q^{-1}), && (4.19)
\end{aligned}
$$

which is both a polynomial of degree n in q^x for $n = 0, 1, \dots, N$ and a polynomial of degree x in $q^{-n} + abq^{n-1}$ for $x = 0, 1, \dots, N$. In (4.19) we have used the notation for *dual q^{-1}-Hahn polynomials* (see [13, §3.7] for dual q-Hahn polynomials; exchange there in [13, (3.7.1)] n and x, replace q by q^{-1}, and next replace γ by qb^{-1} and δ by qa^{-1}).

Formula (4.15) can be rewritten as:

$$
Q_n^{a,b,N;q}(x) = \sum_{k=0}^{\min(n,x)} q^{\frac{1}{2}(n-k)(n-k-1)}\,(-a)^{n-k}\,\frac{(bq^k;q)_{n-k}\,(abq^{n-1};q)_k}{(a;q)_n\,(q;q)_k}\,\frac{(q^n;q^{-1})_k\,(q^x;q^{-1})_k}{(q^N;q^{-1})_k}\,. \tag{4.20}
$$

In notation (4.11) and (2.2) the limit formula [13, (4.6.1)] reads

$$
\lim_{N\to\infty} Q_n^{a,b,N;q}(x) = p_n^{a,b;q}(x). \tag{4.21}
$$

Expressions (4.11), (4.15), (4.20) for q-Hahn polynomials and expressions (4.13) and (4.14) for their weights and dual weights are very similar to expressions (2.2), (2.6), (2.9), (2.4), (2.5), respectively, for little q-Jacobi polynomials, and the q-Hahn expressions immediately turn into their little q-Jacobi counterparts under the limit transition (4.21).

Remark 4.1. Formula (4.11) together with (4.15) is the q-Hahn case of a more general identity observed by Terwilliger [27] related to so-called parameter arrays (see Theorem 4.1 (ii), Lemma 4.2 and Example 5.4 in [27]). The limit for $N \to \infty$ of this q-Hahn case is the little q-Jacobi equality of (2.2) and (2.6). This observation may have relevance for the open problem raised in [27, Problem 11.1].

From (4.11) and (4.15) we see that (3.13) will have meaning with the following substitutions:

$$
y_x := q^x, \quad z_i := q^{-i} + abq^{i-1} \quad (x, i \in \{0, 1, \dots, N\}, \qquad \frac{p_n(y_x)}{p_n(y_N)} := Q_n^{a,b,N;q}(x), \tag{4.22}
$$

where

$$
p_n(y_\nu) = (-a)^{-n}\,q^{-\frac{1}{2}n(n-1)}\,\frac{(a;q)_n}{(b;q)_n}. \tag{4.23}
$$

With these substitutions, formula (3.13) coincides with formula (4.20) if we put

$$\delta_m = \frac{(qa^{-1})^m}{(b;q)_m\,(q;q)_m\,(q^N;q^{-1})_m}\,. \tag{4.24}$$

We can alternatively compute δ_m from (3.17). This yields δ_m as the right-hand side of (4.5) with an additional factor $1/(q^N;q^{-1})_m$, from which we obtain (4.24).

Formula (3.21) can also be specified in the q-Hahn case. Make substitutions as above and, furthermore, put

$$w_x := w_x^{a,b,N;q}, \qquad \omega_n := \big((p_n(y_N)\big)^2 \omega_n^{a,b,N;q}.$$

Then (3.21) yields

$$\begin{aligned} Q_n^{a,b,N;q}(x) = & \sum_{k=\max(n,x)}^{N} q^{\frac{1}{2}(k-x)(k-x-1)}\,(-a)^{k-x} \\ & \times \frac{(q^N;q^{-1})_k\,(abq^{n+k};q)_{N-k}\,(aq^{N-1};q^{-1})_x\,(b;q)_k}{(a;q)_N\,(b;q)_x\,(q;q)_k}\,\frac{(q^k;q^{-1})_n\,(q^k;q^{-1})_x}{(q^N;q^{-1})_n\,(q^N;q^{-1})_x}\,. \end{aligned} \tag{4.25}$$

Formula (4.25) can also be directly reduced to (4.11) by first reversing the direction of summation in (4.25): substitute $k = N - l$. Then we obtain

$$Q_n^{a,b,N;q}(x) = q^{\frac{1}{2}(N-x)(N-x-1)}\,(-a)^{N-x}\,\frac{(bq^x;q)_{N-x}}{(a;q)_{N-x}}\;{}_3\phi_2\left(\begin{matrix} q^{n-N},\,q^{x-N},\,a^{-1}b^{-1}q^{-N-n+1} \\ b^{-1}q^{-N+1},\,q^{-N} \end{matrix};q,q\right). \tag{4.26}$$

Formula (4.26) follows from (4.11) by the transformation formula [9, (3.2.2)]. Note that (4.15) was obtained from (4.11) by a different application of this transformation formula. Also note that the summation reversion changed the upper times lower formula (4.25) into the lower times upper formula (4.26).

The ${}_3\phi_2$ in (4.26) can be written both as a *Hahn polynomial* and a *dual q-Hahn polynomial*: we rewrite (4.26) for $n, x = 0, 1, \ldots, N$ as

$$\begin{aligned} & q^{-\frac{1}{2}(N-x)(N-x-1)}\,(-a)^{x-N}\,\frac{(a;q)_{N-x}}{(bq^x;q)_{N-x}}\,Q_n^{a,b,N;q}(x) \\ & \quad = Q_{N-n}(q^{x-N};b^{-1}q^{-N},a^{-1}q^{-N},N;q) && (4.27) \\ & \quad = R_{N-x}(q^{n-N}+a^{-1}b^{-1}q^{-N-n+1};b^{-1}q^{-N},a^{-1}q^{-N} \mid q). && (4.28) \end{aligned}$$

Thus the left-hand side of the above identities is both a polynomial of degree $N - x$ in $q^{n-N} + a^{-1}b^{-1}q^{-N-n+1}$ and a polynomial of degree $N - n$ in q^{x-N}.

Remark 4.2. Analogous to our observation for (4.15) (see Remark 4.1), formula (4.26) can also be obtained as a consequence of Theorem 4.1 (ii), Lemma 4.2 and Example 5.5 in [27]. Also observe that finite orthogonal polynomial systems whose duals are also orthogonal polynomial systems, the so-called *Leonard pairs*, were extensively studied by Terwilliger, see for instance [25], [26], [27], [28]. Associated with a Leonard pair is a *split decomposition*, which gives rise to a *parameter array*. Formula (10) in [27], which depends on the parameters from that array, is essentially the same as our formula (3.13).

4.3 0-Hahn functions

We obtain as an immediate corollary of (4.20):

Theorem 4.3. *The limit functions* (0-Hahn functions)

$$Q_n^{a,b,N;0}(x) := \lim_{q\downarrow 0} Q_n^{a,b,N;q}(x) \qquad (n, x = 0, 1, \ldots, N) \tag{4.29}$$

exist. They are equal to the little 0*-Jacobi functions* $p_n^{a,b;0}(x)$ *(see* (2.11)–(2.13)*) restricted to* $x = 0, 1, \ldots, N$*:*

$$Q_n^{a,b,N;0}(x) = p_n^{a,b;0}(x) \qquad (x = 0, 1, \ldots, N). \qquad (n, x = 0, 1, \ldots, N) \tag{4.30}$$

From (4.13) and (4.14) we get limits

$$w_x^{a,b,N;0} := \lim_{q\downarrow 0} w_x^{a,b,N;q} \quad = \begin{cases} \dfrac{1-a}{1-ab} & \text{if } x = 0, \\ \dfrac{(1-a)(1-b)}{1-ab}\, a^x & \text{if } 0 < x < N, \\ \dfrac{1-b}{1-ab}\, a^N & \text{if } x = N, \end{cases} \tag{4.31}$$

$$\omega_n^{a,b,N;0} := \lim_{q\downarrow 0} \omega_n^{a,b,N;q} \quad = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{1-a}{a(1-b)} & \text{if } n = 1, \\ \dfrac{(1-a)(1-ab)}{a^n(1-b)} & \text{if } 2 \le n \le N. \end{cases} \tag{4.32}$$

Note that $w_x^{a,b,N;0} > 0$ $(0 \le x \le N)$ and $\omega_n^{a,b,N;0} > 0$ $(0 \le n \le N)$ if $0 < a < 1$ and $b < 1$. They are almost the same as the weights (2.18) and duals weights (2.19) for little 0-Jacobi polynomials. The orthogonality relation (4.12) remains valid for $q = 0$ by taking limits for $q \downarrow 0$.

Remark 4.4. Theorem 4.3 was first stated by Haran [11, (7.3.37)], where the limit functions (4.30) are given in [11, (4.4.10)]. The limit result there follows (although this is not explicitly stated) from the expression [11, (7.3.20), (7.3.21)] for q-Hahn polynomials, which reads in our notation as:

$$Q_n^{a,b,N;q}(x) = q^{nx} \frac{(q^{N-x-n+1};q)_n}{(q^{N-n+1};q)_n} \frac{(b;q)_n}{(a^{-1}q^{1-n};q)_n} \; {}_3\phi_2 \left(\begin{matrix} q^{-n}, a^{-1}q^{1-n}, q^{-x} \\ b, q^{N-x-n+1} \end{matrix} ; q, q \right) \tag{4.33}$$

$$= \sum_{k=0}^{\min(n,x)} q^{\frac{1}{2}(n-k)(n+2x-3k-1)} \frac{(-a)^{n-k}\, (q^{N-x};q^{-1})_{n-k}\, (bq^k;q)_{n-k}}{(q^N;q^{-1})_n\, (a;q)_{n-k}\, (q;q)_k} (q^n;q^{-1})_k\, (q^x;q^{-1})_k\,. \tag{4.34}$$

Formula (4.33) follows from (4.15) by the transformation formula [9, (3.2.5)]. Theorem 4.3 can be obtained from (4.34) by letting $q \downarrow 0$.

Haran's [11, Ch. 7] notation is connected with ours by:

$$\frac{\phi^{(\alpha)\beta}_{q(N),n}(x, N-x)}{\phi^{(\alpha)\beta}_{q(N),n}(N, 0)} = Q_n^{q^\beta, q^\alpha, N; q}(x), \qquad \zeta_{(q)}(s) = ((q^s; q)_\infty)^{-1}. \tag{4.35}$$

5 Interpretation as spherical functions over p-adic spaces

We mention for completeness the overall picture concerning our main object of study, the little q-Jacobi polynomials. Let $\mathbb{F}$ be a local field. $\mathbb{F}$ can be Archimedean ($\mathbb{R}$, $\mathbb{C}$) or non-Archimedean, that is, either a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers or the Laurent series over a finite field (see [23, Chapter 4] for details). For $\mathbb{F}$ non-Archimedean, let $\mathcal{O}$ stand for the ring of integers. Let $K_{\mathbb{F}}$ be the maximal compact subgroup of $GL(d, \mathbb{F})$: the orthogonal group O_d, the unitary group U_d or $GL(d, \mathcal{O})$. The natural representation of $K_{\mathbb{F}}$ arising from its action on the projective space is given by

$$\rho^{\mathbb{F}} : K_{\mathbb{F}} \longrightarrow U\big(L^2(\mathbb{P}^{d-1}_{\mathbb{F}})\big), \qquad [\rho^{\mathbb{F}}(g)f](x) = f(g^{-1}x).$$

This representation admits a multiplicity free decomposition into irreducible representations:

$$L^2(\mathbb{P}^{d-1}_{\mathbb{F}}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{U}^{\mathbb{F}}_n.$$

The label $\mathbb{F}$ on the various objects here emphasizes the dependence on the field. However, the point here is that the decomposition does not depend on the field. Moreover, the irreducibles occurring in the decomposition for fixed n correspond to each other when we go through the various fields (cf. [11], [6]). This correspondence is realized by the observation that, for all $\mathbb{F}$, the little q-Jacobi polynomial of degree n has limits which are spherical functions in $\mathcal{U}^{\mathbb{F}}_n$ for all $\mathbb{F}$'s. The orthogonality measure of these limit functions is the projection of the Haar measure from $K_{\mathbb{F}}$ to the space $\mathbb{P}^{d-1}_{\mathbb{F}} \times_{K_{\mathbb{F}}} \mathbb{P}^{d-1}_{\mathbb{F}}$, on which the spherical functions live.

It also turns out (cf. [6]) that this scheme could be generalized to representations arising from the action of these groups on Grassmannians.

5.1 Interpretation of little 0-Jacobi functions

Let $\mathbb{F}$ be a p-adic field, $\mathcal{O}$ the ring of integers, $\wp$ the maximal ideal in $\mathcal{O}$, p^r the cardinality of the residue field $\mathcal{O}/\wp$ (p a prime number), and $p\mathcal{O} = \wp^e$ (e the ramification index, see again [23, Chapter 4] for details). We look at the representation of $GL(d, \mathcal{O})$, the maximal compact subgroup of $GL(d, \mathbb{F})$, defined by

$$\rho\colon GL(d, \mathcal{O}) \longrightarrow B\left(L^2(\mathbb{P}^{d-1}(\mathbb{F}))\right), \quad [\rho(g)f](x) = f(g^{-1}x),$$

arising from the action of $GL(d, \mathcal{O})$ on $\mathbb{P}^{d-1}(\mathbb{F}) \cong \mathbb{P}^{d-1}(\mathcal{O})$. Let P_m stand for the intersection of a standard maximal parabolic subgroup of type $(m, d-m)$ in $GL(d, \mathbb{F})$ with $GL(d, \mathcal{O})$. In

particular, $\mathbb{P}^{d-1}(\mathcal{O}) \simeq GL(d,\mathcal{O})/P_1$. When we look at P_m-invariants in the representation then we have

$$L^2\left(\mathbb{P}^{d-1}(\mathcal{O})\right)^{P_m} = L^2\big(P_m\backslash GL(d,\mathcal{O})/P_1\big).$$

The group $GL(d,\mathcal{O})$ acts on $\mathbb{P}^{d-1}(\mathcal{O})$ and hence on its quotients $\mathbb{P}^{d-1}(\mathcal{O}/\wp^k)$. Denote the stabilizer of $(1:0:\cdots:0) \in \mathbb{P}^{d-1}(\mathcal{O}/\wp^k)$ in $GL(d,\mathcal{O})$ by $P_1^{(k)}$. The space $\mathbb{P}^{d-1}(\mathcal{O}/\wp^k)$ is the boundary of a ball of radius k in the rooted tree with root valency $\frac{p^{rd}-1}{p^r-1}$ (the cardinality of the projective space over the residue field) and remaining vertices of degree $p^{rd}+1$. The orbits of P_m on $\mathbb{P}^{d-1}(\mathcal{O}/\wp^k)$ consist of $k+1$ points and the orbits of the limit space are parameterized by $\mathbb{Z}_{\geq 0} \cup \{\infty\}$:

$$P_m\backslash GL(n,\mathcal{O})/P_1 = \coprod_{k=0}^{\infty} P_m \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \wp^k & 0 & \cdots & 0 & 1 \end{pmatrix} P_1 \simeq \mathbb{Z}_{\geq 0} \cup \{\infty\}. \tag{5.1}$$

The projection μ_p of the Haar measure to the orbit space is given (see [11]) by

$$\mu_p(\{k\}) = w_k^{a,b;0} \quad (k \in \mathbb{Z}_{\geq 0},\ a = p^{-r(d-m)},\ b = p^{-rm}), \qquad \mu_p(\{\infty\}) = 0, \tag{5.2}$$

where the weights $w_k^{a,b;0}$, explicitly given by (2.18), are limits for $q \downarrow 0$ of the weights for the orthogonality of the little q-Jacobi polynomials. Thus, one obtains an interpretation of these weights on a p-adic space if a, b are as in (5.2). Moreover, for these values of the parameters, the p-adic spherical functions (i.e., fixed vectors in the representation ρ under P_m), are the functions $p_n^{a,b,0}$ given by (2.11)–(2.13). Hence, they are limits of little q-Jacobi polynomials, as was shown in [11].

5.2 Interpretation of little 0-Laguerre functions

For the interpretation of the little q-Laguerre polynomials at $q=0$, we look at the action of $GL(m,\mathcal{O})$ on $\mathcal{O}^m$. The orbits of this action are characterized by the minimal valuation of the entries of a vector. Hence, the orbits are: $\mathcal{O}^m \setminus \wp\mathcal{O}^m$, $\wp\mathcal{O}^m \setminus \wp^2\mathcal{O}^m$, $\cdots$, $\{0\}$. The measure of the orbits is precisely the orthogonality measure for the 0-Laguerre functions $p_n^{p^{-rm},0;0}$:

$$\mu(\wp^j\mathcal{O}^m \setminus \wp^{j+1}\mathcal{O}^m) = (1-p^{-rm})p^{-rjm} \qquad (j \in \mathbb{Z}_{\geq 0}).$$

Moreover, by taking $GL(m,\mathcal{O})$-invariants in the decomposition to irreducibles of the representation $L^2(\mathcal{O}^m)$, we find the 0-Laguerre functions $p_n^{p^{-rm},0;0}$ as fixed vectors. In the special case $m=r=1$ (group of p-adic units acting on the ring of p-adic integers) this interpretation was already obtained by Dunkl and Ramirez [8].

The two pictures (Jacobi and Laguerre) are related in the following manner. One can restrict the action of $GL(d,\mathcal{O})$ on $\mathbb{P}^{d-1}(\mathcal{O})$, to an action of its subgroup $GL(m,\mathcal{O})$ embedded in the top left corner on the subspace $\{(x:0:\cdots:0:1)|\ x \in \mathcal{O}^m\} \subset \mathbb{P}^{d-1}(\mathcal{O})$. This action is clearly the same as the action $GL(m,\mathcal{O})$ on $\mathcal{O}^m$. In terms of the parameters a and b, this restriction amounts to setting $b=0$, thus ignoring the irrelevant part of the space.

5.3 Product formula — p-adic

In this subsection we derive the p-adic product formula. We assume that $a = p^{-r(d-m)}$ and $b = p^{-rm}$. Then the nonnegativity conditions (2.26) for the product formula (2.23) are valid. Let

$$\nu(i) = \sum_{j \geq i} \mu_p(j) = \begin{cases} 1 & \text{if } i = 0, \\ \dfrac{1-b}{1-ab}\, a^i & \text{if } i > 0. \end{cases} \tag{5.3}$$

Our first step is to look at the spherical functions with a different normalization, which makes them idempotents in the convolution algebra $L^1\big(P_m \backslash GL(d, \mathcal{O})/P_1\big)$. We also rewrite them in terms of the measure, rather than in terms of a and b.

$$e_0(x) = \frac{p_0^{a,b;0}(\infty)}{\|p_0^{a,b;0}\|^2}\, p_0^{a,b;0}(x) \quad = \quad 1, \tag{5.4}$$

$$e_n(x) = \frac{p_n^{a,b;0}(\infty)}{\|p_n^{a,b;0}\|^2}\, p_n^{a,b;0}(x) \quad = \left\{ \begin{array}{cl} 0 & \text{if } 0 \leq x < n-1, \\ -\dfrac{1}{\nu(n-1)} & \text{if } x = n-1, \\ \dfrac{1}{\nu(n)} - \dfrac{1}{\nu(n-1)} & \text{if } x > n-1 \end{array} \right\} \quad (n \geq 1). \tag{5.5}$$

Let $\{c_i = \mathbf{1}_{\{i,\ldots,\infty\}}\}$ and $\{g_i = \mathbf{1}_{\{i\}}\}$. For $i \geq 0$ we have:

$$c_i = \sum_{j \geq i} g_i \qquad\qquad g_i = c_i - c_{i+1} \tag{5.6}$$

$$c_i = \nu(i) \sum_{j \leq i} e_j \qquad\qquad e_i = \frac{1}{\nu(i)} c_i - \frac{1}{\nu(i-1)} c_{i-1} \qquad\qquad (c_{-1} := 0) \tag{5.7}$$

The multiplication in the algebra, which is defined by declaring that the e_i's are idempotents, is given by:

$$e_i \star e_j = \delta_{ij} e_i, \tag{5.8}$$

$$c_i \star c_j = \nu(\max\{i,j\}) c_{\min\{i,j\}}, \tag{5.9}$$

$$g_i \star g_j = \begin{cases} \mu(\max\{i,j\}) g_{\min\{i,j\}} & \text{if } i \neq j, \\ (\mu(i) - \nu(i+1)) g_i + \mu(i) \sum_{j>i} g_j & \text{if } i = j. \end{cases} \tag{5.10}$$

At this point we restrict to the p-adic case. In particular we use the fact that the e_i's and g_i's are dual bases in the sense that the former is an idempotent basis for the convolution product, and the latter is an idempotent basis for the pointwise product. The spherical transform intertwines these products and bases. It follows that the multiplication table for the g_i's (after normalizing) with respect to the convolution product is the pointwise multiplication for the idempotents,

giving the desired product formula. If we normalize the g_i's to be orthonormal, by setting $\hat{g}_i = \frac{1}{\mu(i)} g_i$, we get:

$$\hat{g}_i \star \hat{g}_j = \begin{cases} \hat{g}_{\min\{i,j\}} & \text{if } i \neq j, \\ \left(1 - \frac{\nu(i+1)}{\mu(i)}\right)\hat{g}_i + \sum_{j>i} \frac{\mu(j)}{\mu(i)} \hat{g}_j & \text{if } i = j. \end{cases} \tag{5.11}$$

Which agrees with the $c^{a,b,0}_{x,y,z}$ in (2.24).

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